Lagrangian Theory
of
Several-body Systems

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Introduction. Let the N-tuple of 3-vectors $\{\mathbf{x}_i(t) : i = 1, 2, \ldots, N\}$ describe, relative to an inertial frame, the configuration of an N-particle system at time t. To describe the dynamics of the system we would find it natural to introduce the Lagrangian

$$L = \tfrac{1}{2}\sum_i m_i\dot{\mathbf{x}}_i\cdot\dot{\mathbf{x}}_i - U(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \tag{1}$$

Suppose, however, we had elected to describe the particles in terms of their Cartesian relationship to a *reference point* $\mathbf{X}(t)$ *in arbitrarily prescribed motion*: $\mathbf{x}_i = \mathbf{X} + \mathbf{r}_i$. We would then have

$$\begin{aligned} L &= \tfrac{1}{2}\sum_i m_i(\dot{\mathbf{X}} + \dot{\mathbf{r}}_i)\cdot(\dot{\mathbf{X}} + \dot{\mathbf{r}}_i) - U(\mathbf{X} + \mathbf{r}) \\ &= \tfrac{1}{2}M\dot{\mathbf{X}}\cdot\dot{\mathbf{X}} + \dot{\mathbf{X}}\cdot\sum_i m_i\dot{\mathbf{r}}_i + \tfrac{1}{2}\sum_i m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i - U(\mathbf{X} + \mathbf{r}) \end{aligned} \tag{2}$$

in which the dynamical variables—formerly $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$, presently $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$—are still (as before, and not at all surprisingly) N in number. The equations of motion now read

$$m_i\ddot{\mathbf{r}}_i = -m_i\ddot{\mathbf{X}} - \nabla_i U(\mathbf{X} + \mathbf{r}) \qquad (i = 1, 2, \ldots, N) \tag{3}$$

in which we interpret the first term on the right to be a "*fictitious force term*", an artifact of the circumstance that the origin of the $\mathbf{X}$-centered $\mathbf{r}$-frame is (except when $\ddot{\mathbf{X}} = 0$) *non-inertial*. None of which is in any respect problematic.

Now—with an eye to the algebraic simplification of (2)—impose upon the $\mathbf{r}$-variables this *constraint*:

$$\sum_i m_i\mathbf{r}_i = \mathbf{0} \tag{4.1}$$

Equivalently, associate $\mathbf{X}$ with the *center of mass* of the N-particle system:

$$\mathbf{X} = \frac{1}{M}\sum_i m_i\mathbf{x}_i \qquad \text{with} \qquad M = \sum_i m_i \tag{4.2}$$

Equation (2) can now be notated

$$L = \tfrac{1}{2}M\dot{\mathbf{X}}\cdot\dot{\mathbf{X}} + \tfrac{1}{2}\sum_i m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i - U(\mathbf{X}+\mathbf{r}) \tag{5}$$

(note the disappearance of the cross-term), which on its face appears to refer to a system with dynamical variables $\{\mathbf{X},\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_N\}$ more numerous that the variables of the system with which we started. And (5) gives rise to equations of motion which are not only more numerous than but also *inconsistent with* equations (3). What's gone wrong?

In (4) we see that when we subjected the variables $\mathbf{x}_i$ to a cross-term-killing constraint we effectively promoted $\mathbf{X}$ to the status of a *dynamical* variable; it is, according to (4.2), a variable—a *collective* variable—whose t-dependence has now to be extracted from equations of motion, and can no longer be said to be "arbitrarily prescribed." But the Lagrangian (5) is heedless of this circumstance, and gives rise to equations of motion the solutions of which will, in general, stand in violation of (4.1).

To obtain correct results we might, for example, introduce

$$\mathbf{r}_N = \tfrac{1}{m_N}\sum_1^{N-1} m_i\mathbf{r}_i \qquad \text{and} \qquad \dot{\mathbf{r}}_N = \tfrac{1}{m_N}\sum_1^{N-1} m_i\dot{\mathbf{r}}_i$$

into (5) to obtain a Lagrangian of type $L(\dot{\mathbf{X}},\dot{\mathbf{r}}_1,\dots,\dot{\mathbf{r}}_{N-1},\mathbf{X},\mathbf{r}_1,\dots,\mathbf{r}_{N-1})$, but such a procedure bears the formal blemish of a discriminatory asymmetry not natural to the physics of the situation. How to proceed more symmetrically?

In place of (5) write

$$L = \tfrac{1}{2}M\dot{\mathbf{X}}\cdot\dot{\mathbf{X}} + \tfrac{1}{2}\sum_i m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i - U(\mathbf{X}+\mathbf{r}) - \mathbf{g}\cdot\sum_i m_i\mathbf{r}_i \tag{6}$$

where $\mathbf{g}$ is a "Lagrange multiplier" which will be accorded the formal status of a supernumerary dynamical variable. The resulting equations of motion read

$$M\ddot{\mathbf{X}} = -\nabla U \tag{7.1}$$
$$m_i\ddot{\mathbf{r}}_i = -\nabla_i U - \mathbf{g}m_i \quad (i = 1,2,\dots,N) \tag{7.2}$$
$$\mathbf{0} = \textstyle\sum_i m_i\mathbf{r}_i \tag{7.3}$$

where $\nabla = \sum_i \nabla_i$. These are $N+2$ equations of motion in as many variables. The last of these—the "$\mathbf{g}^{\text{th}}$ Lagrange equation of motion" (7.3)—is simply the constraint relation (4.1), and entails $\sum_i m_i \ddot{\mathbf{r}}_i = \mathbf{0}$. Adding equations (7.2) together, and subtracting the result from (7.1), we obtain

$$\mathbf{g} = \ddot{\mathbf{X}} \tag{8}$$

The equations of motion (7) reduce therefore to

$$M\ddot{\mathbf{X}} = -\nabla U \tag{9.1}$$

$$m_i\{\ddot{\mathbf{r}}_i + \ddot{\mathbf{X}}\} = -\nabla_i U \quad (i = 1, 2, \ldots, N) \tag{9.2}$$

which are attractively symmetric (no $\mathbf{r}_i$ has been discriminated against), but redundant: adding equations (9.2) together gives back (9.1). Equations (9.2) are consistent with (3), of which they are a particularized instance. It is instructive to note also that the introduction of (8) into (6) yields a Lagrangian which is distinct from but (by $\ddot{\mathbf{X}} \cdot \sum_i m_i \mathbf{r}_i = \frac{d}{dt}[\dot{\mathbf{X}} \cdot \sum_i m_i \mathbf{r}_i] - \dot{\mathbf{X}} \cdot \sum_i m_i \dot{\mathbf{r}}_i$) gauge equivalent to the Lagrangian of (2).

A simple example: the one-dimensional A_2 molecule. We look now to the Lagrangian theory of what might be called "a one-dimensional A_2 molecule." The constituent "atoms" reside at x_1 and $x_2 > x_1$, and both have mass m. The molecule itself therefore has mass $M = 2m$. We assume the molecule to be bound together by a spring of natural length a and strength k, and to move in an ambient potential $U(x)$. In natural variables the Lagrangian reads

$$L = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - U(x_1) - U(x_2) - \tfrac{1}{2}k[(x_2 - x_1) - a]^2 \tag{10}$$

The variables intuitively most natural to this simple system are the *external* coordinate

$$X = \frac{1}{M}(mx_1 + mx_2) = \tfrac{1}{2}(x_1 + x_2)$$

and the *internal* coordinate $2s = x_2 - x_1$, which describes the instantaneous "length" of the molecule. Immediately

$$\left.\begin{aligned} x_1 &= X - s \\ x_2 &= X + s \end{aligned}\right\} \tag{11}$$

giving

$$L = m(\dot{X}^2 + \dot{s}^2) - U(X - s) - U(X + s) - \tfrac{1}{2}k(2s - a)^2 \tag{13}$$

It becomes analytically advantageous at this point to introduce the variable

$$q = 2s - a$$

which describes molecular length *relative to the rest length* of the molecule;

then $s = \frac{1}{2}(a+q)$ and we have

$$L = m\dot{X}^2 + \tfrac{1}{2}\{\tfrac{1}{2}m\dot{q}^2 - \tfrac{1}{2}Kq^2\} - U(X,q)$$

where $K = 2k$ and

$$\begin{aligned} U(X,q) &= U(X+s) + U(X-s) \\ &= \quad \{U(X) + U'(X)s + \tfrac{1}{2}U''(X)s^2 + \ldots\} \\ &\quad + \{U(X) - U'(X)s + \tfrac{1}{2}U''(X)s^2 + \ldots\} \\ &= 2U(X) + U''(X)s^2 + \ldots \end{aligned}$$

We are led thus to write

$$L = \{m\dot{X}^2 - 2U(X)\} + \tfrac{1}{2}\{\tfrac{1}{2}m\dot{q}^2 - \tfrac{1}{2}Kq^2\} - U_{\text{interaction}}(X,q) \tag{14}$$

with

$$\begin{aligned} U_{\text{interaction}}(X,q) &= \tfrac{1}{4}U''(X)(a^2 + 2aq + q^2) + \ldots \\ &= \tfrac{1}{2}aU''(X)\,q + \ldots \end{aligned}$$

The operative assumption here is that the ambient potential changes little over the dimension of the molecule. The striking absence of a $U'(X)$-term is an artifact of our assumption that $m_1 = m_2 = m$, and means that the interaction is, in leading approximation, "tidal." In this respect the physics of A_2 molecules is distinct from the physics of AB molecules. Looking now to the equations of motion

$$m\ddot{X} = -U'(X) - \tfrac{1}{2}qaU'''(X) \tag{15.1}$$

$$m\ddot{q} + Kq = -aU''(X) \tag{15.2}$$

we find it natural on physical grounds to abandon the 2^{nd} term on the right side of (15.1). Returning with $X(t)$—a solution of the equation thus obtained—to (15.2), we have

$$m\ddot{q} + Kq = \mathcal{F}(t) \qquad \text{with} \qquad \mathcal{F}(t) = -aU''(X(t)) \tag{16}$$

Evidently the "tidal" term on the right side of (15.2) serves in effect to "force" the internal oscillation of the molecule.

The reader who was awaiting the entry of relative variables r_1 and r_2 into the preceeding discussion will have been struck by their absence. Their non-appearance can be attributed to the circumstance that the "asymmetry problem" which motivated our initial discussion does not arise in the case $N = 2$; it is "non-discriminatory" to speak of $s = \frac{1}{2}(r_2 - r_1)$. In this respect the physics of A_n molecules ($n > 2$) is marginally more interesting. The physics even of A_2 molecules becomes markedly more interesting when we give up the one-dimensionality or our problem, for then the molecule can be expected to experience torques, and to "tumble" in ways responsive to the derivative structure of $U(\mathbf{X})$; I shall, however, resist the temptation to enter into an immediate discussion of the details.

A less simple example: the one-dimensional A_3 molecule. We now assume particles of identical mass m to reside at x_1, $x_2 > x_1$ and $x_3 > x_2$ and to be bound by springs identical to those encountered in the previous example. In natural variables the Lagrangian (compare (10)) reads

$$\begin{aligned} L = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - U(x_1) - U(x_2) - U(x_3) \\ - \tfrac{1}{2}k\{[(x_2 - x_1) - a]^2 + [(x_3 - x_2) - a]^2\} \end{aligned} \tag{17}$$

Direct appropriation of (6) gives rise for such a system to

$$\begin{aligned} L = \tfrac{1}{2}M\dot{X}^2 &+ \tfrac{1}{2}m(\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2) - U(X + r_1) - U(X + r_2) - U(X + r_3) \\ &- \tfrac{1}{2}k[(r_2 - r_1) - a]^2 - \tfrac{1}{2}k[(r_3 - r_2) - a]^2 \\ &- g \cdot m(r_1 + r_2 + r_3) \end{aligned} \tag{18}$$

where $M = 3m$ is the mass of the A_3 molecule, and where $X = \frac{1}{3}(x_1 + x_2 + x_3)$ serves to locate its center of mass. Equations (9) acquire therefore this particularized meaning:

$$M\ddot{X} = -U'(X + r_1) - U'(X + r_2) - U'(X + r_3) \tag{19.1}$$

$$\left.\begin{aligned} m(\ddot{r}_1 + \ddot{X}) &= -U'(X + r_1) + k[(r_2 - r_1) - a] \\ m(\ddot{r}_2 + \ddot{X}) &= -U'(X + r_2) - k[(r_2 - r_1) - a] + k[(r_3 - r_2) - a] \\ m(\ddot{r}_3 + \ddot{X}) &= -U'(X + r_3) \qquad\qquad\qquad\quad - k[(r_3 - r_2) - a] \end{aligned}\right\} \tag{19.2}$$

Looking first to (19.1), we observe that

$$\text{RHS of (19.1)} = -3U'(X) - \tfrac{1}{m}U''(X) \cdot \sum_1^3 mr_i - \tfrac{1}{2m}U'''(X) \cdot \sum_1^3 mr_i^2 + \ldots \tag{20}$$

The second term on the right vanishes by definition of the center of mass. It is interesting in this light to notice that the sum encountered in the third term on the right serves to define the instantaneous *moment of inertia relative to the center of mass* (i.e., the centered second moment of the molecular mass distribution), and that the sums encountered in higher-order terms define nameless higher moments of the mass distribution. Upon the abandonment of all such negligible terms, (19.1) reduces to $m\ddot{X} = -U'(X)$: the center of mass moves *as a single atom would move* in the ambient potential.

Looking now to (19.2), we are motivated by the structure of the spring terms to introduce

$$\left.\begin{aligned} q_1 &= (r_2 - r_1) - a \\ q_2 &= (r_3 - r_2) - a \end{aligned}\right\} \tag{21}$$

Inversion of

$$\begin{aligned} q_1 + a &= -r_1 + r_2 \\ q_2 + a &= \qquad\; - r_2 + r_3 \\ 0 &= +r_1 + r_2 + r_3 \end{aligned}$$

gives

$$\left.\begin{aligned} r_1 &= -\tfrac{1}{3}(2q_1 + \ \ q_2) - a\\ r_2 &= +\tfrac{1}{3}(\ \ q_1 - \ \ q_2)\\ r_3 &= +\tfrac{1}{3}(\ \ q_1 + 2q_2) + a \end{aligned}\right\} \tag{22}$$

and when we return with (21) and (22) to (18) we by computation[1] obtain (compare (14))

$$\begin{aligned} L = 3\{\tfrac{1}{2}m\dot{X}^2 - U(X)\} + \{\tfrac{1}{3}m(\dot{q}_1^2 + \dot{q}_1\dot{q}_2 + \dot{q}_2^2) - \tfrac{1}{2}k(q_1^2 + q_2^2)\}\\ - U_{\text{interaction}}(X, q_1, q_2) \end{aligned} \tag{23}$$

where

$$\begin{aligned} U_{\text{interaction}}(X, q_1, q_2) &= \tfrac{1}{2m}U''(X)\cdot m\big[\tfrac{2}{9}a^2 + \tfrac{2}{9}a(q_2 - q_1)\\ &\qquad\qquad + \tfrac{2}{3}(q_1^2 + q_1q_2 + q_2^2)\big] + \dots\\ &= \tfrac{1}{9}\,aU''(X)\cdot(q_2 - q_1) + \dots \end{aligned}$$

To better emphasize the essentials of the situation as it now stands, I write

$$L = L_{\text{center of mass}}(\dot{X}, X) + L_{\text{internal}}(\dot{\mathbf{q}}, \mathbf{q}) + L_{\text{interaction}}(X, \mathbf{q})$$

and assert (on the physical grounds to which I have already alluded, and to a more careful discussion of which I promise to return) that $L_{\text{interaction}}(X, \mathbf{q})$ contributes essentially to the motion of the internal variables $\mathbf{q}$, but inessentially to the motion of X. We are led thus to the equations of motion

$$m\ddot{X} = -U'(X) \tag{24.1}$$

$$\left\{\frac{d}{dt}\frac{\partial}{\partial\dot{q}_i} - \frac{\partial}{\partial q_i}\right\}\mathcal{L} = 0 \qquad (i = 1, 2) \tag{24.2}$$

where $\mathcal{L} = L_{\text{internal}}(\dot{\mathbf{q}}, \mathbf{q}) + L_{\text{interaction}}(X(t), \mathbf{q})$. I have refrained from writing out the detailed implications of (24.2) because those equations are not well-adapted to analytical treatment. To obtain more workable equations we—very

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[1] The tedium of the computation is much reduced if one writes

$$\begin{pmatrix} r_1\\ r_2\\ r_3 \end{pmatrix} = \mathbb{G}\begin{pmatrix} q_1\\ q_2\\ a \end{pmatrix} \text{ and } \begin{pmatrix} \dot{r}_1\\ \dot{r}_2\\ \dot{r}_3 \end{pmatrix} = \mathbb{G}\begin{pmatrix} \dot{q}_1\\ \dot{q}_2\\ 0 \end{pmatrix} \text{ with } \mathbb{G} = \tfrac{1}{3}\begin{pmatrix} -2 & -1 & 1\\ +1 & -1 & 0\\ 1 & 2 & 1 \end{pmatrix}$$

and notices that

$$\mathbb{G}^{\mathrm{T}}\mathbb{G} = \tfrac{1}{9}\begin{pmatrix} 6 & 3 & -1\\ 3 & 6 & 1\\ -1 & 1 & 2 \end{pmatrix}$$

for then $(r_1^2 + r_2^2 + r_3^2)$ and $(\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2)$ become quite easy to evaluate.

much in the spirit of standard small oscillation theory—subject $L_{\text{internal}}(\dot{\mathbf{q}},\mathbf{q})$ to some preparatory massaging.

Writing

$$L_{\text{internal}}(\dot{\mathbf{q}},\mathbf{q}) = \tfrac{1}{6}m\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\end{pmatrix}^{\!\mathrm{T}}\mathbb{M}\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\end{pmatrix} - \tfrac{1}{2}k\begin{pmatrix}q_1\\ q_2\end{pmatrix}^{\!\mathrm{T}}\mathbb{K}\begin{pmatrix}q_1\\ q_2\end{pmatrix}$$

with

$$\mathbb{M} = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \qquad \mathbb{K} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

we introduce new variables (write $\mathbf{q} = \mathbb{R}\,\mathbf{Q}$ with $\mathbb{R}$ a rotation matrix: $\mathbb{R}^{\mathrm{T}}\mathbb{R} = \mathbb{I}$) to obtain

$$L_{\text{internal}}(\dot{\mathbf{Q}},\mathbf{Q}) = \tfrac{1}{6}m\begin{pmatrix}\dot{Q}_1\\ \dot{Q}_2\end{pmatrix}^{\!\mathrm{T}}\mathbb{R}^{\mathrm{T}}\mathbb{M}\,\mathbb{R}\begin{pmatrix}\dot{Q}_1\\ \dot{Q}_2\end{pmatrix} - \tfrac{1}{2}k\begin{pmatrix}Q_1\\ Q_2\end{pmatrix}^{\!\mathrm{T}}\mathbb{R}^{\mathrm{T}}\mathbb{K}\,\mathbb{R}\begin{pmatrix}Q_1\\ Q_2\end{pmatrix} \tag{25}$$

and look to the simultaneous diagonalization of $\mathbb{R}^{\mathrm{T}}\mathbb{M}\,\mathbb{R}$ and $\mathbb{R}^{\mathrm{T}}\mathbb{K}\,\mathbb{R}$. This, of course, is standard "theory of small oscillations" methodology; the only unusual circumstance is that here it is not be spring matrix $\mathbb{K}$ but the mass matrix $\mathbb{M}$ which comes to us in initially non-diagonal form. Some preliminary observations: from $\det(\mathbb{M}-\lambda\mathbb{I}) = \lambda^2 - 4\lambda + 3$ we conclude that the eigenvalues of $\mathbb{M}$ can be described $\lambda = 2 \pm 1$, and that our assignment, therefore, is to discover the matrix

$$\mathbb{R} = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$$

such that

$$\mathbb{R}^{\mathrm{T}}\mathbb{M}\,\mathbb{R} = \begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$$

Our assignment (to say the same thing another way) is to discover the φ such that

$$\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} = \begin{pmatrix}\cos^2\varphi + 3\sin^2\varphi & -2\sin\varphi\cos\varphi\\ -2\sin\varphi\cos\varphi & \sin^2\varphi + 3\cos^2\varphi\end{pmatrix}$$

Immediately $\varphi = -45°$, giving

$$\mathbb{R} = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \tag{26}$$

In any event—quite apart from the details of the argument which led us to (26)—we have only to insert (26) into (25) to obtain

$$L_{\text{internal}}(\dot{\mathbf{Q}},\mathbf{Q}) = \tfrac{1}{6}m(\dot{Q}_1^2 + 3\dot{Q}_2^2) - \tfrac{1}{2}k(Q_1^2 + Q_1^2)$$

The modal motion of the **free** A_3 molecule can therefore be described

$$\ddot{Q}_i + \omega_i^2 Q_i = 0 \tag{27}$$

with

$$\begin{aligned}\omega_1 &= \sqrt{3k/m}\\ \omega_2 &= \sqrt{k/m}\end{aligned}\tag{28}$$

For such a molecule (i.e., for a one-dimensional A_3 molecule in the total absence of an ambient potential) we have

$$\begin{aligned}X(t) &= X_0 + V_0 t\\ q_1(t) &= A_1\cos(\omega_1 t + \delta_1) + A_2\cos(\omega_2 t + \delta_2)\\ q_2(t) &= -A_1\cos(\omega_1 t + \delta_1) + A_2\cos(\omega_2 t + \delta_2)\end{aligned}$$

so (recall (22)) in natural variables we have the following explicit description of the free motion latent in the Lagrangian (17):

$$\begin{aligned}x_1(t) &= X(t) - \tfrac{1}{3}A_1\cos(\omega_1 t + \delta_1) - A_2\cos(\omega_2 t + \delta_2)\\ x_2(t) &= X(t) + \tfrac{2}{3}A_1\cos(\omega_1 t + \delta_1)\\ x_3(t) &= X(t) - \tfrac{1}{3}A_1\cos(\omega_1 t + \delta_1) + A_2\cos(\omega_2 t + \delta_2)\end{aligned}\tag{29}$$

These equations make clear the sense in which the "fast mode" (the mode under the control of A_1) is a "hip-swinging" mode, and the "slow mode" (controlled by A_2) is a "breather mode"—a dance in which m_2 does not participate. So much for the free motion of the system.

Now reinstate the ambient potential $U(x)$. The motion $X(t)$ of the center of mass is now no longer uniform, but accelerated as described (in leading approximation) by (24.1). Of more particular interest is the fact that

$$L_{\text{interaction}} = -\tfrac{1}{9}aU''(X(t))\cdot(q_1 - q_2) = \mathcal{F}(t)\cdot Q_1$$

where $\mathcal{F}(t) = \frac{\sqrt{2}}{9}aU''(X(t))$. The implication is that tidal forces couple (in leading approximation) only to the fast mode. This I find somewhat counterintuitive, since it is the slow "breather" mode of the A_3 molecule which most resembles the solitary mode of the A_2 molecule—a "breather" mode which, as we know from previous work, *does* respond to tidal forces. From

$$L_{\text{molecular}}(\dot{\mathbf{Q}}, \mathbf{Q}, t) = \tfrac{1}{6}m(\dot{Q}_1^2 + 3\dot{Q}_2^2) - \tfrac{1}{2}k(Q_1^2 + Q_1^2) + \mathcal{F}(t)\cdot Q_1$$

we obtain

$$\begin{aligned}\ddot{Q}_1 + \omega_1^2 Q_1 &= \tfrac{3}{m}\mathcal{F}(t)\\ \ddot{Q}_2 + \omega_2^2 Q_2 &= 0\end{aligned}\tag{30}$$

which give back (27) in the absence of tidal forces.

Motion of a "struck" A_3 molecule. In place of (17) we now have

$$L = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \tfrac{1}{2}k\{[(x_2 - x_1) - a]^2 + [(x_3 - x_2) - a]^2\} - x_1 F(t)$$

—the inessential assumption here being that (since objects are most commonly struck on their exposed *surfaces*) it is the end-particle m_1 which has been "struck."